THE EXISTENCE OF THE FUNCTIONS OF THE ELLIPTIC CYLINDER.*

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1. Introduction. The periodic solutions, of period 2π, of the differential equation

$$(1) \qquad \frac{d^2E(\varphi)}{d\varphi^2} + 4\left(\frac{2}{b}\cos 2\varphi + z\right)E(\varphi) = 0, \ b \neq 0,$$

where b and z are real parameters, are known as the functions of the elliptic cylinder.† Heine was the first to attempt to establish‡ the existence of these functions. He tried to show that, for a given value of b, there exist infinitely many real values of z—the characteristic numbers—for each of which (1) has a periodic solution, not identically zero, of period 2π. The method by which he proposed to prove the existence of the characteristic numbers and thus of the functions of the elliptic cylinder is the method used in the present paper. It has already been developed by Dannacher;§ his proof is, however, involved and not evidently rigorous.

An entirely different method of determining the periodic solutions of (1) and establishing their existence was given by Bôcher|| by means of Sturm's Theorem of Oscillation. The existence of the characteristic numbers also follows as a special case of the theory of more general differential equations with periodic coefficients as given, for instance, by Bôcher¶ or Mason.**

Heine's method is, however, historically the first and it is of interest to see how it can be made simple and accurate. Furthermore, this method has furnished a feasible means of computing the actual values†† of the characteristic numbers for a given b.

* Presented to the Society, September 5, 1917.

† The equation (1) is fundamental in physical problems dealing with elliptic membranes and cylinders; see, for example, Mathieu, Journal de Liouville, 2e Serie, vol. 13 (1868), pp. 137–204; Maclaurin, Transactions of the Cambridge Philosophical Society, vol. 17′ (1898), pp. 41–109.

‡ Handbuch der Kugel funktionen I (1878/81), pp. 401–413.

§ Inaugural Dissertation, Zürich, 1906.

|| Ueber die Reihenentwicklungen der Potentialtheorie (1894), p. 181.

¶ Comptes Rendus, vol. 141 (1906), p. 928.

** Comptes Rendus, vol. 141 (1906), p. 1086.

†† Butts, American Journal of Mathematics, vol. 30 (1908), pp. 129–155.

In discussing (1) we restrict $b > 0$; the case $b < 0$ may be transformed into the case $b > 0$ by replacing b by $-b$ and φ by $\varphi + \pi/2$.

2. **The general periodic solution of period 2π.** Assume that (1) has a solution $E(\varphi)$, periodic of period 2π; then $E(\varphi)$ and its first three derivatives are continuous. Hence $E(\varphi)$ may be developed into a Fourier Series

$$E(\varphi) = \tfrac{1}{2}a_0 + \sum_{i=1}^{\infty} (a_i \cos i\varphi + c_i \sin i\varphi), \tag{2}$$

which for all values of φ converges absolutely and uniformly to $E(\varphi)$* and which may be differentiated term by term twice.†

Substituting in (1) the development for $\dfrac{d^2E(\varphi)}{d\varphi^2}$, together with that for $E(\varphi)$, we have

$$\begin{aligned}\sum_{i=1}^{\infty} i^2(a_i \cos i\varphi + c_i \sin i\varphi) \\ = 4\left(\frac{2}{b}\cos 2\varphi + z)(\tfrac{1}{2}a_0 + \sum_{i=1}^{\infty} (a_i \cos i\varphi + c_i \sin i\varphi)\right).\end{aligned} \tag{2'}$$

The series development for $E(\varphi)$ is absolutely convergent and it remains so on multiplication term by term by $[(2/b)\cos 2\varphi + z]$; hence in the resulting series we may remove parentheses and rearrange terms at pleasure. Thus the right hand side of (2′) may be rewritten as

$$4\left\{\frac{a_0}{b}\cos 2\varphi + \frac{1}{b}\sum_{i=1}^{\infty} 2(a_i \cos i\varphi \cos 2\varphi + c_i \sin i\varphi \cos 2\varphi) + \frac{za_0}{2} + z\sum_{i=1}^{\infty}(a_i \cos i\varphi + c_i \sin i\varphi)\right\}$$

and again, by the use of trigonometric identities, as

$$\begin{aligned}\frac{4}{b}\Big\{ a_2 + z\frac{ba_0}{2} + (a_3 + (zb+1)a_1)\cos\varphi \\ + \sum_{i=2}^{\infty}(a_{i-2} + a_{i+2} + zba_i)\cos i\varphi + (c_3 + (zb-1)c_1)\sin\varphi \\ + (c_4 + zbc_2)\sin 2\varphi + \sum_{i=3}^{\infty}(c_{i-2} + c_{i+2} + zbc_i)\sin i\varphi\Big\}.\end{aligned} \tag{2a}$$

We may also rearrange the terms on the left of (2′):

$$\sum_{i=1}^{\infty} i^2 a_i \cos i\varphi + \sum_{i=1}^{\infty} i^2 c_i \sin i\varphi. \tag{2b}$$

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* Bôcher, Introduction to the Theory of Fourier Series, these Annals, Series 2, vol. 7 (1906), p. 109.

† Bôcher, Loc. cit., p. 116.

The series (2a) converges to the same value as the series (2b); hence we may equate corresponding coefficients,* thereby obtaining four groups of relations, one involving the a_{2i}, one the a_{2i+1}, one the c_{2i} and one the c_{2i+1}. Since further we may write

$$E(\varphi) = \frac{a_0}{2} + \sum_{i=1}^{\infty} a_{2i} \cos 2i\varphi + \sum_{i=1}^{\infty} a_{2i-1} \cos (2i - 1)\varphi$$
$$+ \sum_{i=1}^{\infty} c_{2i} \sin 2i\varphi + \sum_{i=1}^{\infty} c_{2i-1} \sin (2i - 1)\varphi,$$

we have

THEOREM 1. *If a periodic solution, of period* 2π, *of the differential equation* (1) *exists, it is a linear combination of four particular solutions of the form*:

$$E_1(\varphi) = \tfrac{1}{2}\alpha_0 + \sum_{i=1}^{\infty} \alpha_i \cos 2i\varphi, \qquad E_2(\varphi) = \sum_{i=1}^{\infty} \beta_i \sin 2i\varphi,$$

$$E_3(\varphi) = \sum_{i=1}^{\infty} \gamma_i \cos (2i - 1)\varphi. \qquad E_4(\varphi) = \sum_{i=1}^{\infty} \delta_i \sin (2i - 1)\varphi.$$

We need therefore to consider only the existence of periodic solutions of these types and it is sufficient to confine our attention to solutions of the first type; the work for the others is quite similar.

3. **The solution** $E_1(\varphi)$.—Suppose that $E_1(\varphi)$, a real, even, periodic function of period π, is a solution of (1). It possesses then the Fourier development

$$E_1(\varphi) = \tfrac{1}{2}\alpha_0 + \sum_{i=1}^{\infty} \alpha_i \cos 2_i\varphi. \tag{3}$$

The formal substitution of this development in (1) yields according to the first of the four groups mentioned above the recurrence relations for determining the α's:

$$\begin{aligned}
\alpha_0 &= 1, \\
\alpha_1 &= -\tfrac{1}{2}bz, \\
\alpha_2 &= b(1 - z)\alpha_1 - 1, \\
\alpha_3 &= b(4 - z)\alpha_2 - \alpha_1, \\
&\cdots\cdots\cdots\cdots \\
\alpha_m &= b((m - 1)^2 - z)\alpha_{m-1} - \alpha_{m-2}, \\
&\cdots\cdots\cdots\cdots
\end{aligned} \tag{4}$$

where, remembering that (1) is homogeneous, we have set $\alpha_0 = 1$.

* Bôcher, loc. cit., p. 151.

If $E_1(\varphi)$ is to be a solution of (1), it is necessary* that α_m, as determined by (4), approach zero as m becomes infinite. To prove this condition also sufficient we establish first two theorems which summarize completely the behavior of the sequence $\alpha_0, \alpha_1, \alpha_2, \cdots$ for a given value of b.

THEOREM 2. *For any particular value of z, real or complex, the sequence $|\alpha_0|, |\alpha_1|, |\alpha_2|, \cdots$ is ultimately either continually increasing or continually decreasing.*

Given a particular value of z; choose μ so that

$$b\,|\,m^2 - z\,| > 2, \qquad m \geqq \mu. \tag{5}$$

If the terms of the sequence $|\alpha_\mu|, |\alpha_{\mu+1}|, |\alpha_{\mu+2}| \cdots$ continually decrease, our theorem is granted. If not, there exists a first α, call it α_k, $k \geqq \mu$, such that $|\alpha_{k-1}| \leqq |\alpha_k|$; then $\alpha_k \neq 0$. From (4) we have

$$\begin{aligned} |\alpha_{k+1}| &\geqq b\,|\,k^2 - z\,|\,|\alpha_k| - |\alpha_{k-1}| \\ &\geqq (b\,|\,k^2 - z\,| - 1)\,|\alpha_k| + |\alpha_k| - |\alpha_{k-1}| > |\alpha_k|. \end{aligned}$$

In like manner $|\alpha_{k+n}| > |\alpha_{k+n-1}|$, $n > 1$, and the sequence of absolute values is ultimately a continually increasing sequence. In fact we have

$$|\alpha_\mu| > |\alpha_{\mu+1}| > \cdots > |\alpha_{k-1}| \leqq |\alpha_k| < |\alpha_{k+1}| < |\alpha_{k+2}| < \cdots. \tag{6a}$$

THEOREM 3. *For any particular value of z, real or complex, the sequence $\alpha_0, \alpha_1, \alpha_2, \cdots$ either approaches zero or becomes infinite.*

If the α's remain finite, $|\alpha_m| < G$ for all m, then α_m approaches zero as m becomes infinite. For, from

$$b(m^2 - z)\alpha_m = \alpha_{m+1} + \alpha_{m-1}$$

it follows that $b\,|\,m^2 - z\,|\,|\alpha_m| < 2G$. If the α's do not remain finite, they become infinite. For, the absolute value sequence does not remain finite and by theorem 2 it is from a certain term on continually increasing.

COROLLARY. *If for a particular value of z, real or complex, $\lim\limits_{m=\infty} \alpha_m = 0$, then*

$$|\alpha_\mu| > |\alpha_{\mu+1}| > |\alpha_{\mu+2}| > \cdots \tag{6b}$$

where μ is determined as in (5).

For, if there exists a $k \geqq \mu$ such that $|\alpha_{k-1}| \leqq |\alpha_k|$, then, as in (6a), $|\alpha_k| < |\alpha_{k+1} < \cdots$, and the sequence becomes infinite.

THEOREM 4. *A necessary and sufficient condition that $E_1(\varphi)$ is a solution of the differential equation* (1) *is that*

$$\lim_{m=\infty} \alpha_m = 0.$$

* Bôcher, loc. cit., p. 151.

The condition has already been proved necessary; to prove it sufficient we assume

$$\lim_{m=\infty} \alpha_m = 0.$$

Then from (4) and (6*b*), we have

$$\begin{aligned} | \alpha_n | &\geqq b | (n+1)^2 - z | | \alpha_{n+1} | - | \alpha_{n+2} | \\ &> (b | (n+1)^2 - z | - 1) | \alpha_{n+1} |, \qquad n \geqq \mu. \end{aligned}$$

From this inequality it can be easily inferred that there exists an l, such that

$$| \alpha_n | > bn^2 | \alpha_{n+1} |, \qquad n > l.$$

Hence the series $\sum_{i=1}^{\infty} \alpha_n$, where the α_n are given by (4), converges absolutely. Therefore the series

$$\tfrac{1}{2}\alpha_0 + \sum_{i=1}^{\infty} \alpha_i \cos 2i\varphi$$

converges uniformly and represents a continuous function $E_1(\varphi)$. If we differentiate term by term, we have

$$-2 \sum_{i=1}^{\infty} i\alpha_i \sin 2i\varphi.$$

This series converges uniformly to $\dfrac{dE_1(\varphi)}{d\varphi}$. For, in the series of the coefficients the test ratio is

$$\frac{n+1}{n} \left| \frac{\alpha_{n+1}}{\alpha_n} \right| < \frac{n+1}{bn^3}, \qquad n > l.$$

Similarly we may differentiate a second time term by term, and show that

$$-4 \sum_{i=1}^{\infty} i^2 \alpha_i \cos 2i\varphi$$

converges uniformly to $\dfrac{d^2E_1(\varphi)}{d\varphi^2}$. Hence the condition is sufficient.

4. **The existence of the ρ_i.**—Those values of z for which

$$\lim_{m=\infty} \alpha_m = 0, \qquad b = b_0,$$

are precisely the characteristic numbers. In order to establish their existence, we shall show that a sequence of suitably chosen roots, one from each of the equations $\alpha_{m+k} = 0$, $k = 1, 2, 3, \cdots$, converges to a limiting value for z and that for such a value and only for such a value

$$\lim_{m=\infty} \alpha_m = 0.$$

The α's as given by (4) are, for a fixed value of b, polynomials in z with the Sturmian properties:

1) α_0 is a constant different from zero,
2) α_m is of the mth degree in z, $\quad m = 1, 2, \cdots,$
3) α_m, α_{m+1} are relatively prime, $\quad m = 1, 2, \cdots,$
3) $\alpha_{m+1}\alpha_{m-1} < 0, \quad$ if $\quad \alpha_m = 0.$

Hence the number of real roots of $\alpha_m = 0$ in the interval $a \leqq z \leqq c$, $\alpha_m(a) \neq 0$, $\alpha_m(c) \neq 0$ is at least $| V(c) - V(a) |$, where $V(z_0)$ denotes the number of variations in sign of the sequence $\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_m$ for $z = z_0$. Since

$$V(-\infty) = 0, \qquad V(\infty) = m,$$

$\alpha_m = 0$ has at least m distinct, real roots and therefore exactly m. For, α_m is of the mth degree; accordingly $V(z)$ must increase by unity each time z increasing passes through a root of $\alpha_m = 0$ and we have the following lemma and theorem:

Sturmian Lemma. The number of real roots of $\alpha_m = 0$ *in the interval*

$$a \leqq z \leqq c, \qquad \alpha_m(a) \neq 0, \qquad \alpha_m(c) \neq 0,$$

is precisely $V(c) - V(a)$.

THEOREM 5. *The m roots of* $\alpha_m = 0$ *are all real and distinct.*

The m roots of $\alpha_m = 0$, $m > 1$, are separated by the $m - 1$ roots of $\alpha_{m-1} = 0$, that is,

THEOREM 6. *Between each two successive roots of* $\alpha_m = 0$, $m > 1$, $\alpha_{m-1} = 0$ *has one and only one root.*

For, suppose that in the interval $z_1 < z < z_2$, where z_1, z_2 are two successive roots of

$$\alpha_m = 0, \qquad m > 1,$$

$\alpha_{m-1} = 0$ has no root; then in the interval $z_1 - \epsilon < z < z_1 + \epsilon$, for ϵ sufficiently small and positive, α_{m-1} maintains its sign while α_m changes sign twice. Thus

$$V(z_2 + \epsilon) - V(z_1 - \epsilon) = 0$$

and a contradiction to the lemma first stated is established. Therefore $\alpha_{m-1} = 0$ has at least one root between z_1 and z_2 and certainly not more than one.*

* From Dannacher's work it seems improbable that he knew of Theorem 6. Had he made use of it, his proof of the existence of the ρ_i would have been much simplified.

If $b \geqq 1$, $V(-2) = 0$; if $0 < b < 1$, $V\left(\frac{-2}{b}\right) = 0$. Hence every root of $\alpha_m = 0$, $m = 1, 2, \cdots$, is greater than -2, $-2/b$, according as $b \geqq 1$, $0 < b < 1$. Number the roots of $\alpha_m = 0$ in natural order beginning with the smallest and denote the ith root, $i \leqq m$, of $\alpha_m = 0$ by $r_{m,\,i}$. Since, by Theorem 6, the $m+1$ roots of $\alpha_{m+1} = 0$ are separated by the m roots of $\alpha_m = 0$, $r_{m+1,\,i} < r_{m,\,i}$; similarly $r_{m+n+1,\,i} < r_{m+n,\,i}$, $n > 1$. Hence

$$r_{m,\,i} > r_{m+1,\,i} > r_{m+2,\,i} > \cdots > -2, \qquad -2/b$$

and the ith roots of $\alpha_m = 0$, $\alpha_{m+1} = 0$, $\cdots$ form a decreasing sequence approaching a limiting value ρ_i.

THEOREM 7. *The ith root, $i \leqq m$, of $\alpha_{m+k} = 0$ decreases as k increases and approaches a limiting value ρ_i as k becomes infinite.**

5. **Heine's "fundamental error."**—It is in showing that the ρ_i are the characteristic numbers, that is, are the values of z for which $\lim_{m=\infty} \alpha_m = 0$, that Heine makes his "fundamental error."† He assumes forthwith that the ρ_i are identical with the roots of the limiting function of the α-sequence. The conditions under which this is true were given later by Hurwitz‡ in the theorem: If the sequence of functions $f_1(z)$, $f_2(z)$, $\cdots$ converges uniformly in a finite domain C of the complex variable z to the function $f(z)$, not identically zero, and $f_i(z)$, $i = 1, 2, \cdots$, behaves like a rational function in the interior of C, then the roots of $f(z)$ in the interior of C are identical with the cluster points of the roots of $f_i(z)$ in the interior of C. We can not apply this theorem directly to the α-sequence, since, as a matter of fact, the α-sequence converges only for isolated values of z. However, we shall show that, if we set

$$(7) \qquad \alpha_1 = b\beta_1, \qquad \alpha_m = b^m(m-1)\,!^2\beta_m, \qquad m > 1,$$

the sequence of β's thus obtained converges and converges uniformly in any finite domain C to a function β not identically zero. The ρ_i which are obviously the cluster points of the roots of the β_i as well as of the roots of the α_i will therefore, by Hurwitz's theorem, be identical with the roots of β. If then we show that $\lim_{m=\infty} \beta_m = 0$ when and only when $\lim_{m=\infty} \alpha_m = 0$, it will follow that the roots of $\beta = 0$, the ρ_i, are the characteristic numbers.

* It is a simple matter to determine, as, for instance, Dannacher has done, the distribution of the ρ_i and to show that for m sufficintly large,

$$(m-1)^2 < \rho_m < (m-1)^2 + \frac{1}{b\{b(2m-3)-1\}} < m^2.$$

† Dannacher, loc. cit., p. 1.

‡ Mathematische Annalen, vol. 33 (1889), pp. 246–249.

6. **The β-sequence.**—Substituting (7) in (4) we have, as the recurrence relations for determining the β's:

$$
\begin{aligned}
\beta_1 &= -\frac{z}{2}, \\
\beta_2 &= (1 - z)\beta_1 - \frac{1}{b^2}, \qquad (8)\\
\beta_{m+1} &= \left(1 - \frac{z}{m^2}\right)\beta_m - \frac{\beta_{m-1}}{b^2(m-1)^2m^2}, \qquad m > 1.
\end{aligned}
$$

We first prove:

THEOREM 8. *A necessary and sufficient condition that*

$$\lim_{m=\infty} \beta_m = 0$$

is that

$$\lim_{m=\infty} \alpha_m = 0.$$

Given

$$\lim_{m=\infty} \alpha_m = 0;$$

then from (7)

$$\lim_{m=\infty} b^m(m-1)!^2\beta_m = 0$$

and

$$\lim_{m=\infty} \beta_m = 0,$$

since, even when $b < 1$, $b^m(m-1)!^2$ becomes infinite with m.

Given

$$\lim_{m=\infty} \beta_m = 0.$$

Suppose

$$\lim_{m=\infty} \alpha_m \neq 0;$$

then the α-sequence becomes infinite and, as we saw in the proof of Theorem 2,

$$|\alpha_{n+1}| \geqq (b\,|n^2 - z| - 1)\,|\alpha_n|, \qquad n \geqq k,$$

where the equality sign holds at most for $n = k$, k determined as in (6a). Restrict k further, if necessary, so that

$$k^2 > |z| + \frac{1}{b}; \tag{9}$$

then

$$|\alpha_{n+1}| \geqq (bn^2 - b\,|z| - 1)\,|\alpha_n|, \qquad n \geqq k.$$

By (7) this relation yields

$$|\beta_{k+1}| \geqq \left(1 - \frac{|z| + 1/b}{k^2}\right)|\beta_k|,$$

$$|\beta_{k+2}| > \left(1 - \frac{|z| + 1/b}{(k+1)^2}\right)|\beta_{k+1}|,$$

$$\cdots\cdots\cdots\cdots$$

$$|\beta_{k+s}| > \left(1 - \frac{|z| + 1/b}{(k+s-1)^2}\right)|\beta_{k+s-1}|,$$

whence, on multiplication,

$$|\beta_{k+s}| > |\beta_k| \prod_{r=k}^{k+s-1}\left(1 - \frac{|z| + 1/b}{r^2}\right) > |\beta_k| \prod_{r=k}^{\infty}\left(1 - \frac{|z| + 1/b}{r^2}\right).$$

For, by (9)

$$0 < \frac{|z| + 1/b}{r^2} < 1, \qquad r \geqq k.$$

Since $\sum_{r=k}^{\infty} \frac{|z| + 1/b}{r^2}$ converges absolutely, the infinite product converges* to a value $c > 0$. Hence $|\beta_{k+s}| > c\,|\beta_k|$, $s = 1, 2, \cdots$. But $|\beta_k| \neq 0$, for $|\alpha_k| \neq 0$; therefore $\lim_{m=\infty} \beta_m \neq 0$ and a contradiction is established.

As a preliminary to the proof that the β-sequence converges uniformly, we prove

THEOREM 9. *In every finite domain C the β-sequence remains uniformly finite.*

Without loss of generality we may restrict C to be a circular region, $|z| < R$, with center in the origin. Given a constant $R > 0$, we have to show that a constant M exists, such that $|\beta_n| < M$ for all $|z| < R$ and for all n, or what is sufficient in this case, that a constant M exists, such that $|\beta_n| < M$ for all $|z| < R$ and for all $n \geqq l$, where l is an integer independent of z. We may take $l = \mu$, where μ is determined so that (5) holds for all z in C. For this purpose it suffices to take

$$\mu^2 > \frac{2}{b} + R. \tag{10}$$

1) For every value of z in C for which $\lim_{m=\infty} \alpha_m = 0$, (6*b*) holds and by (7)

$$|\beta_\mu| > b\mu^2 |\beta_{\mu+1}| > b^2\mu^2(\mu+1)^2 |\beta_{\mu+2}| > \cdots.$$

Since by (10) $b\mu^2 > 2$, $|\beta_\mu| > |\beta_{\mu+1} > \cdots$ and

$$|\beta_m| < |\beta_\mu| \text{ for all } m > \mu. \tag{11}$$

* Osgood, Funktiontheorie 1 (1907), p. 459; 2d ed. (1912), p. 528.

2) We want a corresponding relation for every value of z in C for which the α-sequence becomes infinite. For such a va ue of z (6b):

$$| \alpha_\mu | > | \alpha_{\mu+1} | > \cdots > | \alpha_{k-1} | \leqq | \alpha_k | < | \alpha_{k-1} | < \cdots$$

holds, where the μ, but not the k, is independent of z. Since

$$| \alpha_\mu | > | \alpha_{\mu+1} | > \cdots > | \alpha_{k-1} | ,$$

it follows as in (11) that

$$(a) \qquad | \beta_m | < | \beta_\mu | , \text{ for } m = \mu + 1, \cdots, k - 1.$$

To compare $| \beta_k |$ with $| \beta_\mu |$, we first restrict μ, if actually necessary, so that $\mu^2 > 4$; then, on the substitution of (7) in

$$| \alpha_k | \leqq (b(k-1)^2 + b | z |) | \alpha_{k-1} | + | \alpha_{k-2} | ,$$

we obtain

$$| \beta_k | \leqq \left(1 + \frac{| z |}{(k-1)^2}\right) | \beta_{k-1} | + \frac{1}{b^2(k-1)^2(k-2)^2} | \beta_{k-2} |$$

$$< \left(1 + R + \frac{1}{b^2}\right) | \beta_\mu | .$$

Hence

$$(b) \qquad | \beta_k | < K | \beta_\mu | , \qquad \text{where} \qquad K = 1 + R + \frac{1}{b^2}.$$

To compare $| \beta_m |$, $m > k$, with $| \beta_\mu |$, we compare $| \beta_m |$ first with $| \beta_k |$. From

$$| \alpha_{k-1} | \leqq | \alpha_k | < \cdots ,$$

we have by (4)

$$| \alpha_{n+1} | \leqq (b(n^2 + R) + 1) | \alpha_n | ,$$

where the equality sign holds at most for $n = k$. By (7) this relation yields

$$| \beta_{k+1} | \leqq \left(1 + \frac{R + 1/b}{k^2}\right) | \beta_k | ,$$

$$| \beta_{k+2} | < \left(1 + \frac{R + 1/b}{(k+1)^2}\right) | \beta_{k+1} | ,$$

$$\cdots\cdots\cdots\cdots$$

$$| \beta_{k+s} | < \left(1 + \frac{R + 1/b}{(k+s-1)^2}\right) | \beta_{k+s-1} | ,$$

and, by multiplication,

$$| \beta_{k+s} | < | \beta_k | \sum_{r=k}^{k+s-1} \left(1 + \frac{R + 1/b}{r^2}\right) < | \beta_k | \prod_{r=1}^{\infty} \left(1 + \frac{R + 1/b}{r^2}\right).$$

The infinite product converges to a value $L > 1$; hence

(c) $\quad | \beta_m | < L | \beta_k | < G | \beta_\mu |, \quad m \geqq k + 1, \quad G = KL > 1.$

The three inequalities (a), (b), (c) may evidently be combined in the one inequality

(12) $$| \beta_m | < G | \beta_\mu |, \text{ for all } m > \mu.$$

Since $G > 1$, (12) includes (11) and holds for all $| z | < R$. In C $| \beta_\mu |$ has an upper limit H and hence there exists a constant $M = GH$ such that

(13) $$| \beta_m | < M \text{ for all } m \geqq \mu \text{ and all } | z | < R.$$

With this inequality established it is a simple matter to prove

THEOREM 10. *In every finite domain C the β sequence converges uniformly to a function β, not identically zero.*

We again take C as the circular region $| z | < R$. The β-sequence converges uniformly in C if the series

$$\beta_1 + (\beta_2 - \beta_1) + (\beta_3 - \beta_2) + \cdots$$

converges uniformly in C. The general term of this series is by (8)

$$\beta_{n-1} - \beta_n = \frac{-z}{n^2}\beta_m - \frac{\beta_{n-1}}{b^2(n-1)^2n^2};$$

hence by (13)

$$| \beta_{n+1} - \beta_n | < \frac{1}{n^2}\left\{ R | \beta_n | + \frac{| \beta_{n-1} |}{b^2 | n - 1 |^2} \right\} < \frac{M(R + 1/b^2)}{n^2}, \; n \geqq \mu.$$

The series

$$M\left(R + \frac{1}{b^2}\right)\sum_{n=\mu}^{\infty}\frac{1}{n^2}$$

forms a majorante for the β-series; therefore the β-sequence converges uniformly in C to a function β.

If

$$\beta \equiv 0, \qquad \lim_{n=\infty} \alpha_m \equiv 0;$$

but it is easily shown that for $z = n^2$, n an integer at least as great as μ, the α-sequence becomes infinite; thereby a contradiction is established.

7. **Conclusion.** According to § 5 we may now conclude the existence of the characteristic numbers and their identity with the ρ_i and the roots of $\beta = 0$.

THEOREM 11. *For a given value of b there exist infinitely many characteristic numbers, the cluster points of the roots of the equation $\alpha_m = 0$;*

for each of these infinitely many real values of z, and only for these values, (1) *possesses a periodic solution, not identically zero, of period π, of the form*

$$E_1(\varphi) = \tfrac{1}{2}\alpha_0 + \sum_{i=1}^{\infty} \alpha_i \cos 2i\varphi.$$

A similar theorem holds for periodic solutions of (1) of the forms $E_2(\varphi), E_3(\varphi), E_4(\varphi)$.

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